# *R*-curve effect on strength and reliability of toughened ceramic materials

K. DUAN*, Y.-W. MAI
*Centre for Advanced Materials Technology, Department of Mechanical and Mechatronic Engineering, The University of Sydney, New South Wales 2006, Australia*
B. COTTERELL
*Department of Mechanical and Production Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511*

An improved theoretical analysis is presented for the strength and mechanical reliability of ceramic materials with an *R*-curve characteristic. There is good agreement between the predicted flexural strength distribution of an $Al_2O_3/ZrO_2$ ceramic and experimental data. Unlike the conventional two-parameter Weibull approach, this new analysis is able to predict the non-linearity observed in the lnln $1/(1-P_f)$ versus ln $\sigma_f$ strength distribution curve. Compared to the untoughened Griffith material, the *R*-curve material has higher strength and better strength reliability.

## 1. Introduction

Conventional ceramic materials exhibit a large strength scatter because of their poor crack tolerance and the wide range of sizes and shapes of flaws or defects that are introduced during fabrication and machining processes. The large strength variation has been conveniently explained by many researchers using the weakest link theory (WLT) originally proposed by Weibull [1, 2]. This assumes that the failure of the weakest element among all the elements which comprise an isotropic and statistically homogeneous component would cause the whole component to fail. The simplest form of the WLT for a brittle material of volume, $V$, under a uniaxial tension stress, $\sigma$, can be written as

$$P_f = 1 - \exp\left[-V\left(\frac{\sigma}{\sigma_0}\right)^m\right] \qquad (1)$$

where $P_f$ is the failure probability, $\sigma_0$ is a material parameter, and $m$ is the Weibull modulus which is a measure of the strength scatter. Let $\sigma^* = \sigma_0/V^{1/m}$ and rewrite the above equation in the form of natural logarithm, we have

$$\ln\ln 1/(1-P_f) = m\ln\sigma - m\ln\sigma_* \qquad (2)$$

Thus a straight line with a slope $m$ in the lnln $1/(1-P_f)$ versus $\ln\sigma$ plot is obtained. A high $m$ value gives a narrow strength range and little scatter, i.e. a high strength reliability. Conversely, a low $m$ value means a large variation in strength.

Recent studies on the mechanical properties of advanced ceramics have found that the Weibull modulus is enhanced with increasing fracture toughness. However, it has been recognized that the improvement in the strength reliability comes from the crack resistance, $R$, curve characteristic accompanying the toughening process rather than the higher toughness value alone [3–7]. Assuming the $R$-curve can be described by a power-law equation

$$K_R = Aa^n \qquad (3)$$

Kendall *et al.* [3, 4] and other [5, 6] have shown that the Weibull modulus, $m_R$, of a toughened ceramic can be related to modulus, $m_G$, of the original brittle ceramic by

$$m_R/m_G = 0.5/(0.5-n) \qquad (4)$$

In Equation 3, $K_R$ is the crack resistance, $A$ and $n$ are constants and $a$ is crack length. For most ceramics, $n$ is less than 0.5. Consequently, an increased Weibull modulus, $m_R$, is obtained from Equation 4.

It is noted that the crack resistance, $K_R$, of many toughened ceramics is a function of the crack increment, $\Delta a$, rather than the whole crack length, $a$. Furthermore, $K_R = 0$ when $a = 0$, as required by the power-law function does not seem to agree with experimental data [7–9]. Therefore, the physical basis of Equation 3 is somewhat questionable. Also, the use of Weibull modulus in the WLT implies that the strength distributions of both toughened and untoughened ceramics can be represented by straight lines in the lnln $1/(1-P_f)$ versus $\ln\sigma$ plots. However, recent experimental results, particularly on bend specimens, show that the inert strength distribution curves of the toughened ceramic materials do not obey a straight line relationship [10–12] but are non-linear. It is

* *Present address*: CSIRO Division of Exploration and Mining, P.O. Box 883, Kenmore, Queensland 4069, Australia.

 

suggested that this non-linearity can be explained in terms of the $R$-curve effect on inert strength [11, 12].

In the present paper, a theoretical analysis of the strength reliability of toughened ceramics based on the $R$-curve behaviour is given. Then, the experimental inert strength results of a sintered $Al_2O_3/ZrO_2$ ceramic are discussed and compared to the theory.

## 2. Fracture analysis of *R*-curve effect on strength and reliability of ceramics

From fracture mechanics, the failure of a ceramic component is caused by the unstable propagation of a critical crack or flaw with size $a_i$. Based on the statistical analysis of flaws in a solid of volume $V$, the failure probability, $P_f$, can be related to the applied stress, $\sigma$, and crack density, $q(a)$ by [13]

$$P_f = 1 - \exp\left[\int_V \int_{a_i(\sigma)}^{\infty} q(a)\mathrm{d}a\,\mathrm{d}V\right] \quad (5)$$

This equation can be applied to failure of components originating from surface defects if the volume, $V$, is replaced with the area, $S$, in the integral. Clearly, the strength behaviour of a component is controlled by its flaw distribution, $q(a)$. If $q(a)$ is approximated by a Pareto function, then

$$q(a) = \begin{cases} (\rho_f m/2a_{\min})(a_{\min}/a)^{(m+2)/2}, & a_{\min} \leqslant a \\ 0 & a < a_{\min} \end{cases} \quad (6)$$

where $\rho_f$ is the flaw density and failure probability is given by

$$P_f = 1 - \exp\left\{-\int_V \rho_f[a_{\min}/a_i(\sigma)]^{m/2}\mathrm{d}V\right\} \quad (7)$$

or

$$\ln\ln 1/(1 - P_f) = \ln\int_V \rho_f[a_{\min}/a_i(\sigma)]^{m/2}\mathrm{d}V \quad (8)$$

For truly brittle materials, the critical flaw size, $a_i$, is determined from the Griffith criterion, i.e.

$$a_i = (K_{Ic}^0/Y\sigma)^2 \quad (9)$$

where $K_{Ic}^0$ is the fracture toughness and $Y$ is the geometry correction factor due to the crack. Putting $a_i$ into Equation 8 and integrating for uniaxial tensile and bending stresses over the volume $V$, we recover the Weibull strength distribution, Equation 2. However, for the ceramic materials with an $R$-curve characteristic, the criterion for fracture and hence the failure strength distribution must be reconsidered to include the $R$-curve effects. Let $a_i$ be the initial length of the critical crack which causes the fracture of a ceramic component. Before unstable fracture, a stable crack extension, $\Delta a$, can be obtained because of the existence of a $K_R$-curve in this material. It is assumed that the crack resistance, $K_R$, can be related to the crack extension, $\Delta a$, by a function $K_R(\Delta a)$. Then the failure condition can be written as [14]

$$K_I = K_R(\Delta a) \quad (10a)$$

$$\frac{\mathrm{d}K_I}{\mathrm{d}a} \geqslant \frac{\mathrm{d}K_R(\Delta a)}{\mathrm{d}a} \quad (10b)$$

Because the critical crack sizes are usually much smaller than the specimen dimensions, the applied stress intensity factor $K_I$ can be simplified to

$$K_I = Y\sigma(a_i + \Delta a)^{1/2} \quad (11)$$

Substituting Equation 11 into Equation 10, the initial size of the critical crack can be determined as

$$a_i = \left(\frac{K_R}{Y\sigma}\right)^2 - \Delta a \quad (12a)$$

$$a_i \leqslant \left[Y\sigma\Big/2\left(\frac{\mathrm{d}K_R}{\mathrm{d}a}\right)\right]^2 - \Delta a \quad (12b)$$

Combining the above critical conditions with the distribution of the initial crack sizes in the component given in Equation 6, the failure probability of toughened ceramics with $R$-curve characteristic can be calculated. It can be seen that the shape of the failure strength, $\sigma_f$, distribution curve in the $\ln\ln 1/(1 - P_f)$ versus $\ln\sigma_f$ plot is dependent on the $a_i$ versus $\sigma_f$ relationship which is determined explicitly by the crack resistance function, $K_R(\Delta a)$.

In the case of truly brittle materials, the amount of stable crack growth before final failure is equal to zero. Inserting $\Delta a = 0$ in Equations 10–12 and 8, a linear $\ln\ln 1/(1 - P_f)$ versus $\ln\sigma_f$ relationship is obtained as expected. For those materials whose initial crack size, $a_i$, is related to the toughness, $K_R$, by the power-law Equation 3, the $a_i$ versus $\sigma_f$ relation can be written as

$$a_i = \left(\frac{Y\sigma_f}{A}\right)^{2/(2n-1)} \quad (13)$$

Substituting Equation 13 into Equation 8, we obtain

$$\ln\ln 1/(1 - P_f) = \ln\int_V \left[a_{\min}\left(\frac{A}{Y}\right)^{2/(2n-1)}\right]^{m/2} \times \sigma_f^{m/(1-2n)}\mathrm{d}V \quad (14)$$

For tensile and bending specimens, Equation 14 gives a straight line relationship with an increased slope $m/(1 - 2n)$ in the $\ln\ln 1/(1 - P_f)$ versus $\ln\sigma_f$ plot as reported in previous studies [3, 4, 6]. However, the $a_i$–$\sigma_f$ relationship for many toughened ceramics may not be described by such simple functions as Equation 13. In these cases, Equation 12 has to be solved for $a_i$ as a function of $\sigma_f$. An example for a sintered zirconia-toughened alumina ceramic is given in Section 3.

## 3. Results and discussion

The mechanical properties of a sintered $Al_2O_3/ZrO_2$ (ZTA) ceramic developed by Garvie *et al.* at the CSIRO Division of Materials Science and Technology, Australia, for advanced refractory applications have been measured and studied [11, 12, 15–18]. It is found that this ZTA material exhibits a non-linear stress–strain curve and a pronounced $K_R$-curve characteristic. Experiments have also shown that it suffers mechanical fatigue when subjected to cyclic loading. In this section, the inert strength behaviour of this material is analysed on the basis of the theory in Section 2.

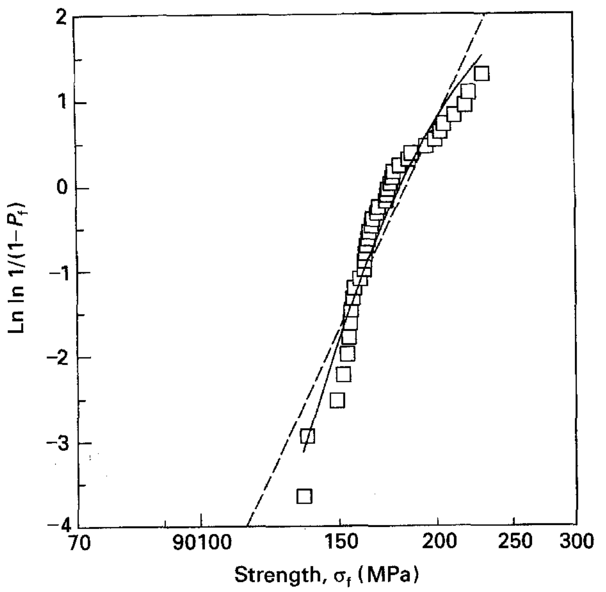


*Figure 1* Flexural strength data of a sintered $Al_2O_3/ZrO_2$ ceramic. (□) Experimental data, (- - -) linear regression, (——) *R*-curve effect.

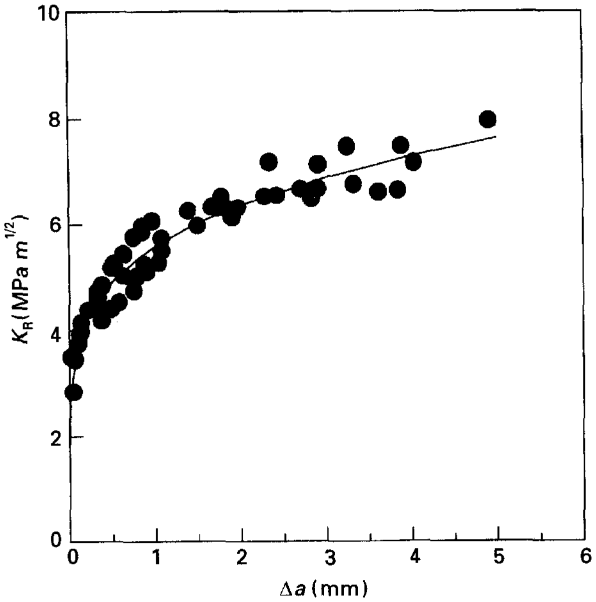


*Figure 2* $K_R$-curve obtained from the CT geometry for the $Al_2O_3/ZrO_2$ ceramic.

The flexural strength was measured by loading plain beams of $\sim 3.8 \times 3.8 \times 43$ mm$^3$ with ground surfaces and chamfered edges along the length in an Instron machine with a fast crosshead speed, 20 mm min$^{-1}$, via a three-point bending jig with a span of 40 mm. The strength data were then plotted against the failure probability in a $\ln\ln 1/(1-P_f)$ versus $\ln \sigma_f$ plot as shown in Fig. 1. The Weibull modulus, *m*, of the ZTA was obtained as 8.44 from the slope of the least squares straight line on the strength data. It should be noted here that the strength data cannot really be described by a straight line relationship of Equation 2 using regression analysis. This effect has also been found in other toughened ceramics with *R*-curve characteristic, e.g. in PSZ/β-$Al_2O_3$ composite [8]. Although the non-linearity can be caused by the shift of the neutral axis in a beam in bending as the crack opens up in the tension half, this effect is believed to be small because the critical crack size is small compared to the beam depth. Here we explain the non-linear Weibull strength distribution in terms of the *R*-curve effect.

It is assumed that the flaw distribution in the volume, *V*, of the sintered $Al_2O_3/ZrO_2$ obeys a Pareto function given by Equation 6 and a unique $K_R$-curve exists when the crack is small compared to the specimen dimensions. Then, its strength distribution can be described by Equation 8 together with the $a_i$ versus $\sigma_f$ relationship determined from the *R*-curve. The $K_R$-curve for short cracks is not available and it is assumed that the initial portion of the $K_R$-curve from the long crack compact tension (CT) geometry as shown in Fig. 2 can be used to obtain the $a_i$ versus $\sigma_f$ relation. This is no doubt a crude assumption but we expect the short-crack $K_R$-curve in the flexural beams, at least for the initial portion, to follow approximately that of the $K_R$-curve in a CT geometry. Of course, for large crack growth, the two $K_R$-curves will deviate from each other. However, this is not relevant here as the short cracks will become unstable in the small crack-growth region in the initial rising portion of the $K_R$-curve. Thus using Equation 12 and Fig. 2 we obtain a numerical solution for $a_i$ as a function of $\sigma_f$, i.e.

$$a_i = 1.028 \times 10^4 + 7.230 \times 10^2 \times \exp(-3.253 \times 10^2 Y \sigma_f) \qquad (15)$$

Combining Equations 8 and 15, we obtain

$$\ln\ln 1/(1-P_f) = \ln \rho_f a_{min}^{m/2} + \ln \int_V [1.028 \times 10^{-4} + 7.230 \times 10^{-2} \exp(-3.253 \times 10^{-2} Y \sigma_f)]^{-m/2} dV \qquad (16)$$

Numerical solutions of Equation 16 for the three-point flexural beam for different $\sigma_f$ with due consideration of the stress gradient across the beam depth give $\ln(V/2)\sigma_f a_{min}^{m/2} = -51.59$, $m = 12.16$ and $Y = 1.7$. The strength distribution curve characterized by these parameters gives much better agreement with the measured data than the two-parameter Weibull function of Equation 2. Note that the Weibull modulus of the ZTA is 12.16 rather than 8.44 with the present analysis. The tensile strength of the material can be predicted by substituting the $a_i$ versus $\sigma_f$ relation given in Equation 15 into Equation 8 so that

$$\ln\ln 1/(1-P_f) = \ln \frac{V}{2} \rho_f a_{min}^{m/2} + \ln[1.028 \times 10^{-4} + 7.230 \times 10^{-2} \exp(-3.253 \times 10^{-2} Y \sigma_f)]^{-m/2} \qquad (17)$$

which is no longer the simple straight line relationship given by Equation 2 because of the *R*-curve effect. If there was no *R*-curve characteristic in this ZTA material and $K_{Ic}^0 = 1.9$ MPa m$^{1/2}$, then combining the Griffith criterion of Equations 9 and 8, the strength distributions of the untoughened material subjected to

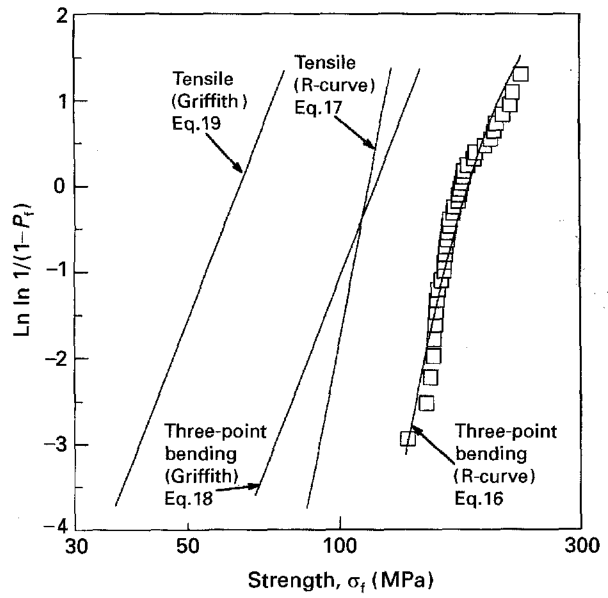


*Figure 3* Comparison of the predicted flexural and tensile strength distributions for the ($R$-curve) toughened and (Griffith) untoughened ceramics. The effects of the $R$-curve characteristic are clearly demonstrated in these curves. The experimental flexural strength data are also given. ($Y = 1.7$, $K_{Ic}^0 = 1.9\,\mathrm{MPa\,m^{1/2}}$, $m = 12.16$, $\ln(V/2\,\rho_f a_{min}^{m/2} = -51.59$; $\rho_f = 10^9$ and $a_{min} = 20\,\mu\mathrm{m}$.)

bending and tensile stresses could also be predicted. Thus the bending strength distribution is given by

$$\ln\ln 1/(1 - P_f) = \ln\frac{V}{2}\rho_f a_{min}^{m/2} + m\ln(Y\sigma/K_{Ic}^0) - 2\ln(m+1) \quad (18)$$

and the tensile strength distribution is

$$\ln\ln 1/(1 - P_f) = \ln\frac{V}{2}\rho_f a_{min}^{m/2} + m\ln(Y\sigma/K_{Ic}^0) \quad (19)$$

The strength distribution curves according to Equations 16–19 and known values of $Y$, $m$, $K_{Ic}^0$ and $\ln[(V/2)\rho_f a_{min}^{m/2}]$ are plotted in Fig. 3. Several comments can be made here. (i) The flexural strength is larger than the tensile strength for a given failure probability. (ii) The effective slope of the strength distribution curves for the ceramic with an $R$-curve characteristic is bigger than that for the untoughened material, i.e. the effective Weibull modulus, $m$, is increased due to the $R$-curve effect. (iii) The non-linearity in the flexural strength distribution curve of the toughened ceramic is consistent with the $R$-curve effect.

## 4. Conclusion

The procedure for analysing the $R$-curve effect on strength and reliability of ceramic materials has been proposed and applied to the experimental results of a toughened $Al_2O_3/ZrO_2$ ceramic. It is found that the proposed technique gives a much better prediction of the flexural strength distribution of the $Al_2O_3/ZrO_2$ ceramic which has an $R$-curve behaviour, than the two-parameter Weibull approach. In particular, the new analysis does accommodate the non-linearity of the strength distribution curve in the $\ln\ln 1/(1 - P_f)$ versus $\ln \sigma_f$ plot. Compared to the untoughened Griffith material the $R$-curve characteristic increases both the strength and strength reliability of the toughened material.


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